Assignment 3

Hand in: Section 6.4, no 10, Supplementary Exercises (1) (2) and (3).

Deadline: Feb 1, 2019.

Section 6.3: no 10b, 11b, 14; Section 6.4: no 9, 10.

Supplementary Exercises

1. Establish the following limits: For $\alpha > 0$,

   (a)
   $$\lim_{x\to\infty} \frac{x^\alpha}{e^x} = 0 \ .$$

   (b)
   $$\lim_{x\to\infty} \frac{\log x}{x^\alpha} = 0 \ .$$

   (c)
   $$\lim_{x\to 0^+} x^\alpha \log x = 0 \ .$$

   Note: (b) and (c) follow from (a).

2. Show that for $x \in [-1/2, 1)$,
   $$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots \ .$$
   Hint: Consider $x \in [0,1)$ and $[-1/2, 0)$ separately.

3. Let
   $$q(x) = -12 + x^2 + 3x^4.$$
   Determine the coefficients in
   $$q(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 \ .$$

4. Let f be infinitely differentiable function. Suppose that there is a polynomial p of degree n such that for some $\delta, C > 0$,
   $$|f(x) - p(x)| \le C|x - x_0|^{n+1} \ , \forall x \in [x_0 - \delta, x_0 + \delta] \ .$$
   Show that p must be the n-th Taylor polynomial of f at x_0.

A Generalized Mean-Value Theorem

A parametric curve, by definition, is simply a continuous map $\gamma = (f_1, f_2, \cdots, f_n)$ from some $[a, b]$ to $\mathbb{R}^n$. We are concerned plane curves $n = 2$ only. It is called a regular parametric curve if further $\gamma' = (f_1', f_2')$ exists and does not vanish, that is, $|\gamma'| = \sqrt{|f_1'|^2 + |f_2'|^2} > 0$ on (a, b).

Generalized Mean-Value Theorem. Let γ be a regular parametric curve on $[a, b]$ on the plane. There exists some $c \in (a, b)$ and $k \neq 0$ such that

$$\gamma(b) - \gamma(a) = k\gamma'(c) \ .$$

Remark 1 Take $\gamma(x) = (x, f(x))$ where f is continuous on $[a, b]$ and differentiable on (a, b). Then γ is a regular parametric curve. By this theorem,

$$(b - a, f(b) - f(a)) = k(1, f'(c)) \ , \quad \text{some } c \in (a, b) \ .$$

That is, $(f(b) - f(a))/(b - a) = f'(c)$, the original Mean-Value Theorem.

Remark 2 In general, looking at each component we have

$$f_1(b) - f_1(a) = kf_1'(c) \ , \quad f_2(b) - f_2(a) = kf_2'(c) \ .$$

When f_1' never vanish on (a, b), we obtain

$$\frac{f_2(b) - f_2(a)}{f_1(b) - f_1(a)} = \frac{f_2'(c)}{f_1'(c)} \ .$$

This is Cauchy Mean-Value Theorem. Our generalized mean-value theorem does not put an extra assumption on f_2, instead it requires the parametric curve be regular.

Sketchy Proof. the Generalized Mean-Value Theorem. WLOG assume $\gamma(a) = (0, 0)$. Rotate the axes so that the vector from $\gamma(a)$ to $\gamma(b)$ become horizontal, that is, from $(0, 0)$ to $(\alpha, 0)$ where $\alpha = (f_1^2(b) + f_2^2(b))^{1/2}$. Let the new curve be γ_1. Now image a horizontal line is dropped from infinity. It will first rest on a point P on the rotated curve $\{\gamma_1(t) : t \in (a, b)\}$ if we assume it is somewhere positive, otherwise replace it by $-\gamma_1$. Parallel to the x-axis, the tangent at $\gamma_1(c)$ is of the form $(\xi, 0)$ for some non-zero ξ. Hence, $(\alpha, 0) = k(\xi, 0)$ where $k = \alpha/\xi$. Rotating back to the original curve, the desired result follows. The interested reader may fill in the details.